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Duality Theorems for Linear Systems and Convex Systems

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We show that, if $(F \to^u X)$ is a linear system, $\Omega \subset X$ a convex target set and $h: X \to \bar{R}$ a convex functional, then, under suitable assumptions, the computation of $\inf h(\{y \in F \mid u(y) \in \Omega\})$ can be reduced to the computation of the infimum of h on certain strips or hyperplanes in F, determined by elements of $u^*(X^*)$, or of the infima on F of Lagrangians, involving elements of $u^*(X^*)$. Also, we prove similar results for a convex system $(F \to^u X)$ and the convex cone Ω of all non-positive elements in X.

1. INTRODUCTION

We recall (see [11]) that a triple $(F \to^u X)$ consisting of two (real) locally convex spaces F, X and a continuous linear mapping u of F into X is called a *locally convex linear system*. If F and X are (real) normed spaces or Banach spaces, then $(F \to^u X)$ is called a *normed linear system*, respectively, a *Banach linear system*. For a locally convex linear system we shall also use, sometimes, the brief term *linear system* (in contrast with the terminology of [4], where a "linear system" means what we call here a Banach linear system). Similarly, we shall call *convex system* any triple $(F \to^u X)$ consisting of a locally convex space F, a partially ordered locally convex space X and a convex mapping u of F into X (i.e., such that $u(\lambda y_1 + (1 - \lambda) y_2) \leqslant \lambda u(y_1) + (1 - \lambda) u(y_2)$ for all $y_1, y_2 \in F$ and all λ with $0 \leqslant \lambda \leqslant 1$).

In Sections 2–4 of the present paper, we shall study the following optimization problem: Given a locally convex linear system $(F \to^u X)$, a convex subset Ω of X with $u(F) \cap \Omega \neq \varnothing$, called *target set*, and a convex functional $h: F \to \bar{R} = [-\infty, +\infty]$, find convenient formulae for

$$a = \inf_{\substack{y \in F \\ u(y) \in \Omega}} h(y), \tag{1.1}$$

under certain suitable assumptions on u, Ω and h. In the particular case when $(F \to^u X)$ is a Banach linear system and h is a finite and continuous convex functional on F, this problem has been studied, applying classical separation theorems to certain convex subsets of X, by Rolewicz ([4, Sect. 5]); for such $(F \to^u X)$ and h, the particular case when $\Omega = \{x_0\}$, a singleton, has been con-

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sidered by Rolewicz ([4, Sect. 3]) and in our paper [11], where we have worked in the space F rather than in X (for the particular case when $h(y) = \|y\|$ for all $y \in F$, see also [3, 10] and the references therein). Let us also observe that in the particular case when $F = X$ and $u = I_F$, the identity mapping, problem (1.1) reduces to the classical convex optimization problem

$$a = \inf_{y \in \Omega} h(y) = \inf h(\Omega), \tag{1.2}$$

called also the problem of solving the convex program $(\Omega, h|_\Omega)$.

In Sections 2–4 of the present paper we shall study the optimization problem (1.1) with a different method, namely, by applying some hyperplane theorems and Lagrangian theorems of duality obtained in our previous papers [7–9]. To this end, it might seem natural to regard problem (1.1) as a particular problem of type (1.2), namely, as

$$a = \inf_{y \in M} h(y) = \inf h(M), \tag{1.3}$$

where M is the convex subset of F defined by

$$M = \{y \in F \mid u(y) \in \Omega\}; \tag{1.4}$$

however, the results obtained via problem (1.3) would yield formulae for (1.1) involving only functionals $\Psi \in F^*$, the set of all continuous linear functionals on F (and in the particular case when Ω is a singleton, only functionals $\Psi \in (\operatorname{Ker} u)^\perp$, i.e., such that $\Psi \in F^*$ and $\Psi(y) = 0$ for all $y \in \operatorname{Ker} u$). Since we want to obtain formulae for (1.1) involving functionals $\Psi \in u^*(X^*)(\subset(\operatorname{Ker} u)^\perp)$, we shall use a different approach. Namely, embedding problem (1.1) in the family of perturbed optimization problems (with the usual conventions $\inf \varnothing = +\infty$, $\sup \varnothing = -\infty$, to be used throughout this paper)

$$a_x = \inf_{\substack{y \in F \\ u(y) \in x + \Omega}} h(y) \qquad (x \in X), \tag{1.5}$$

we shall apply our hyperplane theorems and Lagrangion duality theorems to bhe associated primal functional $f\colon x \to a_x$ defined by (1.5); in the particular case when $\Omega = \{x_0\}$, a singleton, we shall also use a slightly different primal functional.

We recall now some of our previous hyperplane theorems and Lagrangian theorems of duality of [7–9], on problem (1.2), which we shall use as main tools in the present paper.

THEOREM 1.1 [8, Corollary 2.1]. *Let E be a locally convex space and $f\colon E \to \bar{R} = [-\infty, +\infty]$ a lower semi-continuous convex functional. Then*

$$f(x_0) = \sup_{0 \neq \Phi \in E^*} \inf_{\substack{x \in E \\ \Phi(x) = \Phi(x_0)}} f(x) \qquad (x_0 \in E). \tag{1.6}$$

THEOREM 1.2 [9, Lemma 2.1; 7, Theorem 2.1]. *Let E be a locally convex space, $f: E \to \bar{R}$ a proper convex functional, and $x_0 \in E$.*

(a) *If $f(x_0) = f^{**}(x_0)$ (in particular, if f is lower semi-continuous on E), then*

$$f(x_0) = \sup_{\Phi \in E^*} \inf_{x \in E} \{f(x) + \Phi(x) - \Phi(x_0)\}. \tag{1.7}$$

(b) *If f is also finite and continuous on E, then for any convex subset G of E*

$$\inf f(G) = \sup_{\Phi \in E^*} \inf_{x \in E} \{f(x) + \Phi(x) - \sup \Phi(G)\}, \tag{1.8}$$

and there exists $\Phi_0 \in E^*$ such that

$$\inf f(G) = \inf_{x \in E} \{f(x) + \Phi(x) - \sup \Phi(G)\} \tag{1.9}$$

(*i.e., for which the* sup *in* (1.8) *is attained*).

As was observed in [7], the latter result remains valid if we assume only that the proper convex functional f is finite and continuous at one point of G.

THEOREM 1.3 [8, Corollary 3.2, Proposition 3.1]. *Let E be a locally convex space and f a finite and continuous convex functional on E. Then*

(a) *We have* (1.6) *and for each $x_0 \in E$ there exists $\Phi_0 \in E^*$, $\Phi_0 \neq 0$, such that*

$$f(x_0) = \inf_{\substack{x \in E \\ \Phi_0(x) = \Phi_0(x_0)}} f(x) \tag{1.10}$$

(*i.e., for which the* sup *in* (1.6) *is attained*).

(b) *When $f(x_0) > \inf f(E)$, for a functional $\Phi_0 \in E^*$, $\Phi_0 \neq 0$, we have* (1.10) *if and only if there exists $\alpha_0 \in R$ such that*

$$f(x_0) = \inf_{x \in E} \{f(x) + \alpha_0 \Phi_0(x) - \alpha_0 \Phi_0(x_0)\} \tag{1.11}$$

(*i.e., such that the* sup *in* (1.7) *is attained for $\Phi = \alpha_0 \Phi_0$*).

Actually, we shall show in this paper that Theorem 1.3(a) remains also valid under weaker assumptions and we shall also use that sharpening of Theorem 1.3(a).

Let us describe now briefly the contents of each section of the paper.

In Section 2 we shall obtain for the optimization problem (1.1) some "strip theorems" of weak duality, stating, roughly speaking, that under certain assumptions we have

$$\inf_{\substack{y\in F\\ u(y)\in\Omega}} h(y) = \sup_{0\neq\Phi\in X^*} \inf_{\substack{y\in F\\ \Phi(u(y))\in\Phi(\Omega)}} h(y); \tag{1.12}$$

thus, these results reduce the computation of (1.1) to that of the infimum of h on certain "strips" $\{y\in F \mid u^*(\Phi)(y)\in\Phi(\Omega)\}$ (they are theorems "of weak duality," since the sup in (1.12) need not be attained). Of course, in the particular case when $\Omega=\{x_0\}$, a singleton, these strip theorems become "hyperplane theorems" of weak duality, which reduce the computation of (1.1) to that of the infimum of h on certain hyperplanes in F. In the particular case when $F=X$ and $u=I_F$, the identity mapping, we shall obtain some strip theorems for the classical convex optimization problem (1.2) (again, when Ω is a singleton, these reduce to hyperplane theorems); we shall see that these strip theorems not only complement the main hyperplane theorem (of strong duality) for problem (1.2), given in our previous paper [6], but they also imply a new hyperplane theorem of weak duality for that problem.

In Section 3 we shall obtain for the optimization problem (1.1) some Lagrangian theorems of weak duality, stating that under certain assumptions we have

$$\inf_{\substack{y\in F\\ u(y)\in\Omega}} h(y) = \sup_{\Phi\in X^*} \inf_{y\in F} \{h(y)+\Phi(u(y))-\sup\Phi(\Omega)\}; \tag{1.13}$$

in the case when $F=X$, $u=I_F$ and Ω is a singleton, we have called the results of this type "quasi-Lagrangian" duality theorems, in our previous paper [9] (since the sup in (1.13) need not be attained), but here we shall abandon that terminology and use the more customary term: Lagrangian theorems of "weak" duality.

In Section 4 we shall first give a sharpening of Theorem 1.3, mentioned above, and then we shall use it to obtain for the optimization problem (1.1) some strip theorems (which, when Ω is a singleton, reduce to hyperplane theorems) and Lagrangian theorems of strong duality, i.e., of the form (1.12) or (1.13), with the sup being attained for some $\Phi=\Phi_0$.

As shown by the above, in Sections 2–4 of the present paper it will turn out that not only the general hyperplane theorems and Lagrangian duality theorems for the problem of convex optimization (1.2) can be applied to yield new (as well as known) results on problem (1.1), but things work also in the opposite direction, so actually there is a continuous interaction between the two theories.

In Section 5 we shall give some counter-examples related to the assumptions (continuity, semi-continuity, openness, etc.) and to the attainment of the suprema in the right-hand sides, in the results mentioned above.

Finally, in Section 6, applying our methods of Sections 2–4 to convex systems $(F\to^u X)$ and to the convex cone $\Omega=\{x\in X \mid x\leqslant 0\}$, of all non-positive elements in X, we shall obtain some strip theorems and Lagrangian theorems of duality for the optimization problem (1.1) for such systems, with (1.12) and (1.13) replaced, respectively, by

$$\inf_{\substack{y\in F\\ u(y)\leqslant 0}} h(y) = \sup_{0<\Phi\in X^*} \inf_{\substack{y\in F\\ \Phi(u(y))\leqslant 0}} h(y), \tag{1.14}$$

$$\inf_{\substack{y\in F\\ u(y)\leqslant 0}} h(y) = \sup_{0\leqslant\Phi\in X^*} \inf_{\substack{y\in F\\ \Phi(u(y))\leqslant 0}} \{h(y) + \Phi(u(y))\}; \tag{1.15}$$

these results may be regarded as complements to the usual Kuhn–Tucker theorem.

We shall consider in this paper only *real* spaces F, X, E, since the extension to complex scalars can be obtained with the usual methods. We shall use the standard terminology. Let us mention that by "hyperplane" we shall always mean: closed hyperplane.

2. Strip Theorems and Hyperplane Theorems of Weak Duality for Linear Systems

LEMMA 2.1. *Let F, X be two linear spaces, $u: F \to X$ a linear mapping, Ω a convex subset of X with $u(F) \cap \Omega \neq \varnothing$, and $h: F \to \bar{R} = [-\infty, +\infty]$ a convex functional. Then the functional $f: X \to \bar{R}$, defined by*

$$f(x) = \inf_{\substack{y\in F\\ u(y)\in x+\Omega}} h(y) \qquad (x \in X) \tag{2.1}$$

is convex and $\operatorname{dom}(f) = \{x \in X \mid f(x) < +\infty\} \subset u(F) - \Omega$.

Proof. Let $x_1, x_2 \in X$ and $0 \leqslant \lambda \leqslant 1$, and let $\epsilon > 0$. Then, by (2.1), there exist $y_i \in F$ and $\omega_i \in \Omega$ with $u(y_i) = x_i + \omega_i$, such that $h(y_i) \leqslant f(x_i) + \epsilon$ $(i = 1, 2)$; but then, since u is linear and Ω is convex, $y = \lambda y_1 + (1 - \lambda) y_2 \in F$ satisfies $u(y) \in \lambda x_1 + (1 - \lambda) x_2 + \Omega$, whence, by (2.1) and since h is convex, we obtain

$$\begin{aligned} f(\lambda x_1 + (1 - \lambda) x_2) &\leqslant h(\lambda y_1 + (1 - \lambda) y_2) \\ &\leqslant \lambda h(y_1) + (1 - \lambda) h(y_2) \leqslant \lambda f(x_1) + (1 - \lambda) f(x_2) + \epsilon, \end{aligned}$$

which, since $\epsilon > 0$ was arbitrary, proves that f is convex. Finally, we have $\operatorname{dom}(f) \subset u(F) - \Omega$; indeed, if $x \in X \backslash (u(F) - \Omega)$, then $\{y \in F \mid u(y) \in x + \Omega\} = \varnothing$, whence, by (2.1), $f(x) = \inf \varnothing = +\infty$, that is, $x \in X \backslash \operatorname{dom}(f)$. This completes the proof of Lemma 2.1.

LEMMA 2.2. *Let F, X be two locally convex spaces, $u: F \to X$ a mapping, Ω a subset of X with $u(F) \cap \Omega \neq \varnothing$, and $h: F \to \bar{R}$ a functional, such that Ω and the (possibly empty) sets*

$$\Gamma_r = u(\{y \in F \mid h(y) \leqslant r\}) \qquad (r \in R) \tag{2.2}$$

are closed for a topology τ on X, weaker than or equal to the initial topology on X, and either Ω or the sets Γ_r $(r \in R)$ are compact for τ. Then the functional $f: X \to \bar{R}$, defined by (2.1), *is lower semi-continuous.*

Proof. Let $r \in R$ and let $\{x_\delta\}_{\delta \in \Delta}$ be a generalized sequence in

$$S_r = \{x \in X \mid f(x) \leqslant r\}, \tag{2.3}$$

and assume that $x_\delta \to x_0$ (in the initial topology). Then, given $\epsilon > 0$, by $x_\delta \in S_r$ and (2.1) there exist $y_\delta \in F$ with $u(y_\delta) \in x_\delta + \Omega$, such that $h(y_\delta) \leqslant r + \epsilon$ $(\delta \in \Delta)$. Thus, $u(y_\delta) \in \Gamma_{r+\epsilon}$, so $x_\delta \in u(y_\delta) - \Omega \subset \Gamma_{r+\epsilon} - \Omega$ $(\delta \in \Delta)$ and $x_\delta \to x_0$, whence, by our topological assumptions on Ω and $\Gamma_{r+\epsilon}$, it follows that $x_0 \in \Gamma_{r+\epsilon} - \Omega$. Consequently, there exists $y_\epsilon \in F$ with $h(y_\epsilon) \leqslant r + \epsilon$, such that $x_0 \in u(y_\epsilon) - \Omega$. Therefore, by (2.1), $f(x_0) \leqslant h(y_\epsilon) \leqslant r + \epsilon$, whence, since $\epsilon > 0$ was arbitrary, $f(x_0) \leqslant r$, so S_r is closed. This completes the proof of Lemma 2.2.

THEOREM 2.1. *Let $(F \to^u X)$ be a locally convex linear system, Ω a convex subset of X with $u(F) \cap \Omega \neq \varnothing$, and $h: F \to \bar{R}$ a convex functional, such that Ω and the sets Γ_r $(r \in R)$ defined by* (2.2) *are closed for a topology τ on X, weaker than or equal to the initial topology on X, and either Ω or the sets Γ_r $(r \in R)$ are compact for τ. Then we have*

$$\inf_{\substack{y \in F \\ u(y) \in \Omega}} h(y) = \sup_{0 \neq \Phi \in X^*} \inf_{\substack{y \in F \\ \Phi(u(y)) \in \Phi(\Omega)}} h(y). \tag{2.4}$$

Proof. Define a functional $f: X \to \bar{R}$ by (2.1). Then, by Lemma 2.1, Lemma 2.2, and Theorem 1.1, we obtain

$$\begin{aligned} \inf_{\substack{y \in F \\ u(y) \in \Omega}} h(y) &= f(0) = \sup_{0 \neq \Phi \in X^*} \inf_{\substack{x \in X \\ \Phi(x) = 0}} f(x) \\ &\leqslant \sup_{0 \neq \Phi \in X^*} \inf_{\substack{x \in u(F) - \Omega \\ \Phi(x) = 0}} f(x) = \sup_{0 \neq \Phi \in X^*} \inf_{\substack{x \in u(F) - \Omega \\ \Phi(x) = 0}} \inf_{\substack{y \in F \\ u(y) \in x + \Omega}} h(y). \end{aligned} \tag{2.5}$$

We shall show that for each $\Phi \in X^*$ we have

$$\bigcup_{\substack{x \in u(F) - \Omega \\ \Phi(x) = 0}} \{y \in F \mid u(y) \in x + \Omega\} = \{y \in F \mid \Phi(u(y)) \in \Phi(\Omega)\}, \tag{2.6}$$

which, together with (2.5) and the obvious inequality $\geqslant$ in (2.4), will complete the proof. Indeed, if $x \in u(F) - \Omega$, $\Phi(x) = 0$, $y \in F$, $u(y) \in x + \Omega$, then $\Phi(u(y)) \in \Phi(x + \Omega) = \Phi(\Omega)$. Conversely, if $y \in F$, $\Phi(u(y)) = \Phi(x')$, where $x' \in \Omega$, then for $x = u(y) - x' \in u(F) - \Omega$ we have $\Phi(x) = \Phi(u(y)) - \Phi(x') = 0$ and $u(y) = x + x' \in x + \Omega$, which proves (2.6). This completes the proof of Theorem 2.1.

Remark 2.1. (a) In particular case when $(F \to^u X)$ is a Banach linear system and h is a finite and clntinuous convex functional on F, Theorem 2.1 has been proved, with a different argument, by Rolewicz ([4, Theorem 5.1]; however, the assumption $u(F) \cap \Omega \neq \varnothing$ should be added there).

(b) As shown by the above proof of Lemma 2.2, it is enough to assume only that the sets $\Gamma_r - \Omega$ $(r \in R)$ are closed in the initial topology.

(c) The sup in (2.4) need not be attained, even when $F = X$, $u = I_F$, and Ω is a singleton (see [8] and Example 5.1 below). Hence, the sup in the other results of this section (except Remark 2.4(c)) need not be attained.

COROLLARY 2.1. *Under the assumptions of Theorem* 2.1, *we have*

$$\inf_{\substack{y \in F \\ u(y) \in \Omega}} h(y) = \sup_{0 \neq \Phi \in X^*} \inf_{\substack{y \in F \\ \Phi(u(y)) \in \Phi(u(F) \cap \Omega)}} h(y). \tag{2.4'}$$

Proof. By the obvious inequality $\geqslant$ in (2.4) (applied to $u(F) \cap \Omega$ instead of Ω) and by Theorem 2.1 we have

$$\begin{aligned} \inf_{\substack{y \in F \\ u(y) \in \Omega}} h(y) &= \inf_{\substack{y \in F \\ u(y) \in u(F) \cap \Omega}} h(y) \geqslant \sup_{0 \neq \Phi \in X^*} \inf_{\substack{y \in F \\ \Phi(u(y)) \in \Phi(u(F) \cap \Omega)}} h(y) \\ &\geqslant \sup_{0 \neq \Phi \in X^*} \inf_{\substack{y \in F \\ \Phi(u(y)) \in \Phi(\Omega)}} h(y) = \inf_{\substack{y \in F \\ u(y) \in \Omega}} h(y), \end{aligned}$$

whence (2.4'), which completes the proof of Corollary 2.1.

In the particular case when $\Omega = \{x_0\}$, a singleton (and hence compact), from Theorem 2.1 we obtain the following result (which, in the case when $(F \to^u X)$ is a Banach linear system and h is a finite and continuous convex functional on F, has been proved, with a different argument, by Rolewicz in [4, Theorem 2.1]):

COROLLARY 2.2. *Let $(F \to^u X)$ be a locally convex linear system, let $x_0 \in u(F)$ and let $h: F \to \bar{R}$ be a convex functional, such that the sets Γ_r defined by* (2.2) *are closed (possibly empty). Then*

$$\inf_{\substack{y \in F \\ u(y) = x_0}} h(y) = \sup_{0 \neq \Phi \in X^*} \inf_{\substack{y \in F \\ \Phi(u(y)) = \Phi(x_0)}} h(y). \tag{2.7}$$

Remark 2.2. In this particular case, the functional $f\colon X \to \bar{R}$ defined in Lemma 2.1 (by formula (2.1)) becomes

$$f(x) = \inf_{\substack{y \in F \\ u(y)=x+x_0}} h(y) \qquad (x \in X), \tag{2.8}$$

which, since $x_0 \in u(F)$, say $x_0 = u(y_0)$, can be written in the form

$$f(x) = \inf_{\substack{y' \in F \\ u(y')=x}} h(y' + y_0) \qquad (x \in X). \tag{2.9}$$

Naturally, in this case one can also consider, instead of f, the usual primal functional

$$f_0(x) = \inf_{\substack{y \in F \\ u(y)=x}} h(y) \qquad (x \in X); \tag{2.10}$$

then, once the lower semi-continuity of f_0 is established (similarly to the argument used in the above proof of Lemma 2.2), by Theorem 1.1 we obtain

$$\begin{aligned}\inf_{\substack{y \in F \\ u(y)=x_0}} h(y) = f_0(x_0) &= \sup_{0 \neq \Phi \in X^*} \inf_{\substack{x \in X \\ \Phi(x)=\Phi(x_0)}} f_0(x) \\ &= \sup_{0 \neq \Phi \in X^*} \inf_{\substack{x \in X \\ \Phi(x)=\Phi(x_0)}} \inf_{\substack{y \in F \\ \Phi(u(y))=\Phi(x)}} h(y) = \sup_{0 \neq \Phi \in X^*} \inf_{\substack{y \in F \\ \Phi(u(y))=\Phi(x_0)}} h(y),\end{aligned}$$

that is, (2.7). The essential advantage of the primal functional f_0 is the fact that its definition (2.10) does not depend on $\Omega = \{x_0\}$ (while the definition (2.8) of f does). Let us note that the primal functional f_0 may be also regarded as the particular case $\Omega = \{0\}$ of (2.1) or as the case $x_0 = 0$ of (2.8).

Remark 2.3. In the particular case when $F = X$ and $u = I_F$, Corollary 2.2 above yields again Theorem 1.1.

It is worth while to mention separately the particular case $F = X$, $u = I_F$ of Theorem 2.1. In this case, denoting $F = X = E$, $\Omega = G$, $h = f$ in Theorem 2.1, we obtain

THEOREM 2.2. *Let E be a locally convex space, G a convex subset of E and $f\colon E \to \bar{R}$ a convex functional, such that G and the (possibly empty) level sets*

$$S_r = \{x \in E \mid f(x) \leqslant r\} \qquad (r \in R) \tag{2.11}$$

are closed for a topology τ on E, weaker than or equal to the initial topology on E, and either G or the sets S_r $(r \in R)$ are compact for τ. Then

$$\inf f(G) = \sup_{0 \neq \Phi \in E^*} \inf_{\substack{x \in E \\ \Phi(x) \in \Phi(G)}} f(x). \tag{2.12}$$

Remark 2.4. (a) For $\Phi \neq 0$, let us call *strip generated by* Φ any set of the form $\{x \in E \mid \Phi(x) \in \langle\alpha, \beta\rangle\} = \Phi^{-1}(\langle\alpha, \beta\rangle)$, where $\langle\alpha, \beta\rangle$ is an interval (possibly closed or open from the left or from the right, possibly infinite). Then, geometrically, the conclusion (2.12) of Theorem 2.2 means that

$$\inf f(G) = \sup_{0 \neq \Phi \in E^*} \inf f(B_\Phi), \tag{2.13}$$

where B_Φ denotes the smallest strip generated by Φ, containing G. Indeed, if $\Phi \in E^*$, $\Phi \neq 0$, then, since G is convex, $\Phi(G) \subset R$ is an interval $\langle\alpha, \beta\rangle$ as above and

$$\{x \in E \mid \Phi(x) \in \Phi(G)\} = B_\Phi \tag{2.14}$$

(since $x \in B_\Phi$ if and only if $H_{\Phi,x} \cap G \neq \varnothing$, where $H_{\Phi,x} = \{z \in E \mid \Phi(z) = \Phi(x)\}$).

(b) Under the assumptions of Theorem 2.2, we have also

$$\inf f(G) = \sup_{\substack{B \in \mathscr{B} \\ B \supset G}} \inf f(B), \tag{2.15}$$

where $\mathscr{B}$ denotes the collection of all strips in E. Indeed, by (2.13), we have clearly the inequality $\leqslant$ in (2.15); on the other hand, assuming that $<$ holds in (2.15), there would exist $B_0 = \{x \in E \mid \Phi_0(x) \in \langle\alpha_0, \beta_0\rangle\} \in \mathscr{B}$ with $B_0 \supset G$, such that $\inf f(G) < \inf f(B_0) \leqslant \inf f(B_{\Phi_0})$, in contradiction with (2.13). This proves (2.15).

(c) It is interesting to compare Theorem 2.2 and Remark 2.4(a) with the following main hyperplane theorem of [6] on the infimum of f on a convex subset G of E ([6, Theorem 2.1 and Remark 2.2(c)]): *Let E be a locally convex space, f a finite and continuous convex functional on E and G a convex subset of E, satisfying*

$$\inf f(E) < \inf f(G) \tag{2.16}$$

and let $\tilde{x}$ be any element of E such that

$$f(\tilde{x}) < \inf f(G). \tag{2.17}$$

Then we have

$$\inf f(G) = \sup_{\substack{0 \neq \Phi \in E^* \\ \sup \Phi(G) \leqslant \Phi(\tilde{x})}} \inf_{\substack{x \in E \\ \Phi(x) = \sup \Phi(G)}} f(x), \tag{2.18}$$

and there exists $\Phi_0 \in E^*$ with $\Phi_0 \neq 0$, $\sup \Phi_0(G) \leqslant \Phi_0(\tilde{x})$, such that

$$\inf f(G) = \inf_{\substack{x \in E \\ \Phi_0(x) = \sup \Phi_0(G)}} f(x) \tag{2.19}$$

(*i.e., for which the* sup *in* (2.18) *is attained*); or, in geometric interpretation,

$$\inf f(G) = \sup_{H \in \mathscr{H}_{G,\tilde{x}}} \inf f(H), \tag{2.20}$$

where $\mathscr{H}_{G,\tilde{x}}$ denotes the collection of all hyperplanes H in E which support G (we use this term in the sense of [6], i.e., not requiring that $H \cap G \neq \varnothing$) and separate G and $\tilde{x}$, and there exists $H_0 \in \mathscr{H}_{G,\tilde{x}}$ such that

$$\inf f(G) = \inf f(H_0) \tag{2.21}$$

(i.e., for which the sup in (2.20) is attained).

In the particular case when G is a linear manifold, say $G = x_0 + S = \{x_0 + s \mid s \in S\}$, where $x_0 \in E$ and S is a linear subspace of E with $\bar{S} \neq E$, for any $\Phi \in E^*$ with $\Phi \neq 0$ we have, clearly,

$$\begin{aligned} \sup \Phi(G) &= \Phi(x_0) && \text{if} \quad \Phi \in S^\perp \\ &= +\infty && \text{if} \quad \Phi \in E^* \backslash S^\perp, \\ \inf \Phi(G) &= \Phi(x_0) && \text{if} \quad \Phi \in S^\perp \\ &= -\infty && \text{if} \quad \Phi \in E^* \backslash S^\perp, \end{aligned}$$

whence,

$$\begin{aligned} \{x \in E \mid \Phi(x) \in \Phi(G)\} &= \{x \in E \mid \Phi(x) = \Phi(x_0)\} && \text{if} \quad \Phi \in S^\perp \\ &= E && \text{if} \quad \Phi \in E^* \backslash S^\perp, \end{aligned} \tag{2.22}$$

where we use the notation

$$S^\perp = \{\Psi \in E^* \mid \Psi(s) = 0 \ (s \in S)\}; \tag{2.23}$$

geometrically, (2.22) means that the smallest strip B_Φ in the direction Φ, containing the linear manifold $G = x_0 + S$, is either a hyperplane containing G, or the whole space E. Consequently, from Theorem 2.2 we obtain

COROLLARY 2.3. *Let E be a locally convex space, $G = x_0 + S$, where $x_0 \in E$ and S is a linear subspace of E with $\bar{S} \neq E$, such that G is closed for a topology τ on E, weaker than or equal to the initial topology on E, and let $f \colon E \to \bar{R}$ be a convex functional, such that the level sets S_r ($r \in R$) defined by* (2.11) *are compact for τ. Then*

$$\inf f(G) = \sup_{0 \neq \Phi \in S^\perp} \inf_{\substack{x \in E \\ \Phi(x) = \Phi(x_0)}} f(x). \tag{2.24}$$

From this corollary we deduce, in particular:

COROLLARY 2.4. *Let E be a reflexive Banach space, $G = x_0 + S$, where $x_0 \in E$ and S is a proper closed linear subspace of E and let $f: E \to \bar{R}$ be a lower semi-continuous convex functional, such that the level sets S_r $(r \in R)$ defined by* (2.11) *are bounded. Then we have* (2.24).

Proof. Since G is a closed linear subspace of E, it is also weakly closed. Furthermore, since the level sets S_r $(r \in R)$ are closed and convex, they are also weakly closed (see, e.g., [1, p. 422, Theorem 13]). But, by our assumption the sets S_r $(r \in R)$ are also bounded and hence, since E is a reflexive Banach space, the sets S_r are weakly compact. Consequently, by Corollary 2.3 (with $\tau =$ the weak topology on E), we have (2.24), which completes the proof.

Remark 2.5. Geometrically, the conclusion (2.24) of Corollaries 2.3 and 2.4 means that

$$\inf f(G) = \sup_{\substack{H \in \mathscr{H} \\ H \supset G}} \inf f(H), \tag{2.25}$$

where $\mathscr{H}$ denotes the collection of all hyperplanes in E.

Combining the assumptions of Theorem 2.2 with the assumptions (2.16), (2.17) of Remark 2.4(c), we shall prove now a strip theorem, which will imply, as corollaries, a result equivalent to the strip theorem 2.2 and a new hyperplane theorem.

THEOREM 2.3. *Let E be a locally convex space, G a convex subset of E and $f: E \to \bar{R}$ a convex functional such that G and, for each $r > \inf f(E)$, the (non-empty) level sets S_r defined by* (2.11) *are closed for a topology τ on E, weaker than or equal to the initial topology on E, and either G or the sets S_r $(r > \inf f(E))$ are compact for τ. Also, assume that* (2.16) *holds and let $\tilde{x} \in E$ be any element satisfying* (2.17). *Then*

$$\inf f(G) = \sup_{\substack{0 \neq \Phi \in E^* \\ \sup \Phi(G) \leqslant \Phi(\tilde{x})}} \inf_{\substack{x \in E \\ \Phi(x) \in \overline{\Phi(G)}}} f(x). \tag{2.26}$$

Proof. Since G is closed for τ, it is also closed for the initial topology and, by (2.17), we have $\tilde{x} \in E \backslash G$. Hence, by a well known separation theorem (see, e.g., [1, p. 418, Corollary 12]), there exists $\Phi \in E^*$, $\Phi \neq 0$, such that

$$\sup \Phi(G) \leqslant \Phi(\tilde{x}) \tag{2.27}$$

(even such that $\sup \Phi(G) < \Phi(\tilde{x})$). Clearly, for any such Φ we have

$$\inf f(G) \geqslant \inf_{\substack{x \in E \\ \Phi(x) \in \overline{\Phi(G)}}} f(x) \tag{2.28}$$

and hence

$$\inf f(G) \geqslant \sup_{\substack{0 \neq \Phi \in E^* \\ \sup \Phi(G) \leqslant \Phi(\tilde{x})}} \inf_{\substack{x \in E \\ \Phi(x) \in \overline{\Phi(G)}}} f(x). \tag{2.29}$$

Assume now, a contrarion, that the inequality in (2.29) is strict, so there exists $\epsilon > 0$ such that

$$\inf f(G) - \epsilon > \sup_{\substack{0 \neq \Phi \in E^* \\ \sup \Phi(G) \leqslant \Phi(\tilde{x})}} \inf_{\substack{x \in E \\ \Phi(x) \in \overline{\Phi(G)}}} f(x); \tag{2.30}$$

by (2.17), we may also assume that

$$\inf f(G) - \epsilon > f(\tilde{x}). \tag{2.31}$$

Case 1°. $\inf f(G) < +\infty$. Let

$$D = \{x \in E \mid f(x) \leqslant \inf f(G) - \epsilon\} = S_{\inf f(G) - \epsilon} . \tag{2.32}$$

Then, by (2.30), $D \neq \varnothing$ and, clearly, $G \cap D = \varnothing$; also, G and D are convex. Furthermore, by (2.30), $\inf f(G) - \epsilon > \inf f(E)$ and thus, by our assumption, G and $D = S_{\inf f(G) - \epsilon}$ are closed for τ and one of them is compact for τ. Hence, by a well-known separation theorem (see, e.g., [1, p. 417, Theorem 10]), there exists $\Phi_0 \in (E, \tau)^* \subset E^*$, $\Phi_0 \neq 0$, such that

$$\inf \Phi_0(D) > \sup \Phi_0(G). \tag{2.33}$$

Consequently,

$$\inf_{\substack{x \in E \\ \Phi_0(x) \in \overline{\Phi_0(G)}}} f(x) \geqslant \inf f(G) - \epsilon; \tag{2.34}$$

indeed, otherwise there would exist $x_0 \in E$ with $\Phi_0(x_0) \in \overline{\Phi_0(G)}$ (hence $\Phi_0(x_0) \leqslant \sup \overline{\Phi_0(G)} = \sup \Phi_0(G)$) such that $f(x_0) < \inf f(G) - \epsilon$ (so, $x_0 \in D$), in contradiction with (2.33). But, by (2.31), we have $\tilde{x} \in D$, whence, by (2.33), $\Phi_0(\tilde{x}) > \sup \Phi_0(G)$, which, together with (2.34), contradicts (2.30). Thus, in (2.29) the equality sign must hold, that is, we have (2.26).

Case 2°. $\inf f(G) = +\infty$. Let

$$D_1 = \{x \in E \mid f(x) \leqslant c\} = S_c , \tag{2.35}$$

where $c \in R$ is any number such that

$$c > \sup_{\substack{0 \neq \Phi \in E^* \\ \sup \Phi(G) \leqslant \Phi(\tilde{x})}} \inf_{\substack{x \in E \\ \Phi(x) \in \overline{\Phi(G)}}} f(x) \qquad (\geqslant -\infty). \tag{2.36}$$

Then, by (2.36), $D_1 \neq \varnothing$ and, by $\inf f(G) = +\infty$, we have $G \cap D_1 = \varnothing$, so the above argument yields (2.34) with $\inf f(G) - \epsilon$ replaced by c, which contradicts (2.36). This completes the proof of Theorem 2.3.

Remark 2.6. (a) If we assume also that τ is stronger than or equal to the weak topology $\sigma(E, E^*)$ (hence $(E, \tau)^* = E^*$), then, in the case when G is τ-compact, we have $\overline{\Phi(G)} = \Phi(G)$ $(\Phi \in E^*)$, so we obtain a particular case of Corollary 2.5 below.

(b) Geometrically, the conclusion (2.26) of Theorem 2.3 means that in (2.13) it is enough to take the supremum over all Φ with $0 \neq \Phi \in E^*$ which separate G and $\tilde{x}$, provided that we take the infimum of f over $\bar{B}_\Phi$ (instead of B_Φ) or, what is the same thing, over the smallest closed strip generated by Φ, containing G.

COROLLARY 2.5. *Under the assumptions of Theorem* 2.3, *we have*

$$\inf f(G) = \sup_{\substack{0 \neq \Phi \in E^* \\ \sup \Phi(G) \leqslant \Phi(\tilde{x})}} \inf_{\substack{x \in E \\ \Phi(x) \in \Phi(G)}} f(x). \tag{2.37}$$

Proof. We can write

$$\begin{aligned}\inf f(G) &\geqslant \sup_{\substack{0 \neq \Phi \in E^* \\ \sup \Phi(G) \leqslant \Phi(\tilde{x})}} \inf_{\substack{x \in E \\ \Phi(x) \in \Phi(G)}} f(x) \\ &\geqslant \sup_{\substack{0 \neq \Phi \in E^* \\ \sup \Phi(G) \leqslant \Phi(\tilde{x})}} \inf_{\substack{x \in E \\ \Phi(x) \in \overline{\Phi(G)}}} f(x) = \inf f(G),\end{aligned}$$

(where the first two inequalities are obvious and the last equality holds by Theorem 2.3), whence (2.37) follows.

Remark 2.7. (a) Corollary 2.5 can be also deduced from Theorem 2.2, by the following argument, which is perhaps more revealing: By Theorem 2.2, we have (2.12). Now, if $\Phi \in E^*$, $\Phi \neq 0$, satisfies $\Phi(\tilde{x}) \in \Phi(G)$, then, taking (by (2.17)) any $c \in R$ with $f(\tilde{x}) < c < \inf f(G)$, we obtain

$$\inf_{\substack{x \in E \\ \Phi(x) \in \Phi(G)}} f(x) \leqslant f(\tilde{x}) < c < \inf f(G); \tag{2.38}$$

therefore, by (2.12),

$$\inf f(G) = \sup_{\substack{0 \neq \Phi \in E^* \\ \Phi(\tilde{x}) \notin \Phi(G)}} \inf_{\substack{x \in E \\ \Phi(x) \in \Phi(G)}} f(x). \tag{2.39}$$

On the other hand if $0 \neq \Phi \in E^*$ and $\Phi(\tilde{x}) \notin \Phi(G)$, then, since $\Phi(G)$ is an interval $\langle\alpha, \beta\rangle$ (by the convexity of G), we have either $\sup \Phi(G) \leqslant \Phi(\tilde{x})$ or $\inf \Phi(G) \geqslant \Phi(\tilde{x})$, which is equivalent to $\sup(-\Phi(G) \leqslant (-\Phi)(\tilde{x})$. Hence, by (2.39), we obtain

$$\inf f(G) \leqslant \sup_{\substack{0 \neq \Phi \in E^* \\ \sup \Phi(G) \leqslant \Phi(\tilde{x})}} \inf_{\substack{x \in E \\ \Phi(x) \in \Phi(G)}} f(x), \tag{2.40}$$

which, together with the obvious opposite inequality, yields (2.37). This completes the proof of Corollary 2.5.

(b) The converse of (a) is also true, that is, Corollary 2.5 implies Theorem 2.2 (and thus, they are equivalent to each other). Indeed, if Corollary 2.5 holds and if we have (2.16), (2.17), then

$$\begin{aligned} \inf f(G) &\geqslant \sup_{0 \neq \Phi \in E^*} \inf_{\substack{x \in E \\ \Phi(x) \in \Phi(G)}} f(x) \\ &\geqslant \sup_{\substack{0 \neq \Phi \in E^* \\ \sup \Phi(G) \leqslant \Phi(\tilde{x})}} \inf_{\substack{x \in E \\ \Phi(x) \in \Phi(G)}} f(x) = \inf f(G), \end{aligned}$$

whence (2.12) follows. On the other hand, if (2.16) does not hold, that is, $\inf f(E) = \inf f(G)$, then for each $\Phi_0 \in E^*$, $\Phi_0 \neq 0$, we have

$$\inf f(G) \geqslant \inf_{\substack{x \in E \\ \Phi_0(x) \in \Phi_0(G)}} f(x) \geqslant \inf f(E) = \inf f(G),$$

whence, again, (2.12) follows.

(c) Geometrically, formulae (2.39) and (2.37) mean that in (2.13) it is enough to take the supremum over all Φ with $0 \neq \Phi \in E^*$ such that $\tilde{x} \notin B_\Phi$ or over all Φ with $0 \neq \Phi \in E^*$ which separate G and $\tilde{x}$.

(d) Under the assumptions of Theorem 2.3, we have also

$$\inf f(G) = \sup_{B \in \mathscr{B}_{G,\tilde{x}}} \inf f(B), \tag{2.41}$$

where $\mathscr{B}_{G,\tilde{x}}$ denotes the collection of all strips in E, containing G and not containing $\tilde{x}$. Indeed, by (2.39), we have clearly the inequality $\leqslant$ in (2.41). On the other hand, assuming that $<$ holds in (2.41), there would exist $B_0 = \{x \in E \mid \Phi_0(x) \in \langle\alpha, \beta\rangle\} \in \mathscr{B}_{G,\tilde{x}}$ such that $\inf f(G) < \inf f(B_0) \leqslant \inf f(B_{\Phi_0})$, which, since $\tilde{x} \notin B_{\Phi_0}$ (because $\tilde{x} \notin B_0$ and $B_{\Phi_0} \subset B_0$), contradicts (2.39).

The above results not only complement the main hyperplane theorem of [6], mentioned in Remark 2.4(c) above, but also yield the following new hyperplane theorem:

COROLLARY 2.6. *Under the assumptions of Theorem* 2.3, *we have* (2.18).

Proof. By the separation theorem (see the beginning of the proof of Theorem 2.3) there exists $\Phi \in E^*$, $\Phi \neq 0$, satisfying (2.27). But, by [6, Lemma

2.1] (which remains valid, with a similar proof,[1] for convex functionals $f: E \to \bar{R} = [-\infty, +\infty]$), for any such Φ we have

$$\inf f(G) \geqslant \inf_{\substack{x \in E \\ \Phi(x) = \sup \Phi(G)}} f(x). \tag{2.42}$$

Consequently,

$$\begin{aligned} \inf f(G) &\geqslant \sup_{\substack{0 \neq \Phi \in E^* \\ \sup \Phi(G) \leqslant \Phi(\tilde{x})}} \inf_{\substack{x \in E \\ \Phi(x) = \sup \Phi(G)}} f(x) \\ &\geqslant \sup_{\substack{0 \neq \Phi \in E^* \\ \sup \Phi(G) \leqslant \Phi(\tilde{x})}} \inf_{\substack{x \in E \\ \Phi(x) \in \overline{\Phi(G)}}} f(x) = \inf f(G) \end{aligned}$$

(where the last equality holds by Theorem 2.3), whence (2.18) follows. This completes the proof of Corollary 2.6.

Remark 2.8. The geometric interpretation of the conclusion (2.18) of Corollary 2.6 is given by (2.20).

3. Lagrangian Theorems of Weak Duality for Linear Systems

Lemma 3.1. *Let F be a set, X a linear space, $u: F \to X$ a mapping, Ω a subset of X with $u(F) \cap \Omega \neq \varnothing$, $h: F \to \bar{R}$ a functional and*

$$f(x) = \inf_{\substack{y \in F \\ u(y) \in x + \Omega}} h(y) \qquad (x \in X). \tag{3.1}$$

Then for any linear functional $\Phi: X \to R$, $\Phi \neq 0$, we have

$$\inf_{x \in X} \{f(x) + \Phi(x)\} = \inf_{y \in F} \{h(y) + \Phi(u(y)) - \sup \Phi(\Omega)\}. \tag{3.2}$$

Proof. Let $y \in F$, $\omega \in \Omega$ and let $x_{y,\omega} = u(y) - \omega \in u(F) - \Omega$. Then $u(y) \in x_{y,\omega} + \Omega$, whence

$$\begin{aligned} \inf_{x \in X} \{f(x) + \Phi(x)\} &\leqslant f(x_{y,\omega}) + \Phi(x_{y,\omega}) \\ &= \inf_{\substack{y' \in F \\ u(y') \in x_{y,\omega} + \Omega}} h(y') + \Phi(x_{y,\omega}) \\ &\leqslant h(y) + \Phi(x_{y,\omega}) = h(y) + \Phi(u(y)) - \Phi(\omega). \end{aligned}$$

Hence, since $y \in F$ and $\omega \in \Omega$ were arbitrary, we obtain

$$\inf_{x \in X} \{f(x) + \Phi(x)\} \leqslant \inf_{y \in F} \{h(y) + \Phi(u(y)) - \sup \Phi(\Omega)\}. \tag{3.3}$$

[1] Indeed, when $\inf f(G) = -\infty$, take any $g \in G$ with $f(g) < +\infty$ and repeat the argument of [6].

Assume now, a contrario, that the inequality in (3.3) is strict, so there exists $x_0 \in X$ such that

$$\inf_{\substack{y\in F\\ u(y)\in x_0+\Omega}} h(y) + \Phi(x_0) = f(x_0) + \Phi(x_0)$$
$$< \inf_{y\in F} \{h(y) + \Phi(u(y)) - \sup \Phi(\Omega)\}.$$

Then there exist $y_0 \in F$, $\omega_0 \in \Omega$ with $u(y_0) = x_0 + \omega_0$, such that

$$h(y_0) + \Phi(x_0) < \inf_{y\in F} \{h(y) + \Phi(u(y)) - \sup \Phi(\Omega)\}$$
$$\leqslant h(y_0) + \Phi(u(y_0)) - \Phi(\omega_0) = h(y_0) + \Phi(x_0),$$

which is impossible. Thus, in (3.3) the equality sign must hold, that is, we have (3.2), which completes the proof of Lemma 3.1.

Combining Lemma 3.1 and Theorem 1.2(a), we obtain

THEOREM 3.1. *Let $(F \to^u X)$ be a locally convex linear system, Ω a convex subset of X with $u(F) \cap \Omega \neq \varnothing$ and $h: F \to (-\infty, +\infty]$ a convex functional with $\inf h(F) > -\infty$, such that Ω and the sets Γ_r $(r \in R)$ defined by* (2.2) *are closed for a topology τ on X, weaker than or equal to the initial topology on X, and either Ω or the sets Γ_r $(r \in R)$ are compact for τ. Then*

$$\inf_{\substack{y\in F\\ u(y)\in\Omega}} h(y) = \sup_{\Phi\in X^*} \inf_{y\in F} \{h(y) + \Phi(u(y)) - \sup \Phi(\Omega)\}. \tag{3.4}$$

Proof. If $h \equiv +\infty$, then (3.4) is obviously true. Assume now that $h \not\equiv +\infty$ and define a functional $f: X \to \bar{R} = [-\infty, +\infty]$ by (3.1). Then, by our assumptions on h, we have $f(x) > -\infty$ $(x \in X)$ and $f \not\equiv +\infty$, that is, f is proper. Furthermore, by Lemma 2.1 and Lemma 2.2, f is convex and lower semicontinuous. Hence, by Theorem 1.2(a) and Lemma 3.1, we obtain

$$\inf_{\substack{y\in F\\ u(y)\in\Omega}} h(y) = f(0) = \sup_{\Phi\in X^*} \inf_{x\in X} \{f(x) + \Phi(x)\}$$
$$= \sup_{\Phi\in X^*} \inf_{y\in F} \{h(y) + \Phi(u(y)) - \sup \Phi(\Omega)\},$$

that is, (3.4), which completes the proof of Theorem 3.1.

COROLLARY 3.1. *Under the assumptions of Theorem* 3.1, *we have*

$$\inf_{\substack{y\in F\\ u(y)\in\Omega}} h(y) = \sup_{\Phi\in X^*} \inf_{y\in F} \{h(y) + \Phi(u(y)) - \sup \Phi(u(F) \cap \Omega)\}. \tag{3.4'}$$

Proof. By the obvious inequality $\geqslant$ in (3.4) (applied to $u(F) \cap \Omega$ instead of Ω) and by Theorem 3.1 we have

$$\inf_{\substack{y\in F\\ u(y)\in\Omega}} h(y) = \inf_{\substack{y\in F\\ u(y)\in u(F)\cap\Omega}} h(y) \geqslant \sup_{\Phi\in X^*} \inf_{y\in F} \{h(y) + \Phi(u(y)) - \sup \Phi(u(F)\cap\Omega)\}$$
$$\geqslant \sup_{\Phi\in X^*} \inf_{y\in F} \{h(y) + \Phi(u(y)) - \sup \Phi(\Omega)\} = \inf_{\substack{y\in F\\ u(y)\in\Omega}} h(y),$$

whence (3.4′), which completes the proof of Corollary 3.1.

It is worth while to mention separately the particular case $F = X$, $u = I_F$ of Theorem 3.1. In this case, denoting $F = X = E$, $\Omega = G$, $h = f$ in Theorem 3.1, we obtain

THEOREM 3.2. *Let E be a locally convex space, G a convex subset of E and $f\colon E \to (-\infty, +\infty]$ a convex functional with $\inf f(E) > -\infty$, such that G and the level sets S_r $(r \in R)$ defined by* (2.11) *are closed for a topology τ on E, weaker than or equal to the initial topology on E, and either G or the sets S_r $(r \in R)$ are compact for τ. Then*

$$\inf f(G) = \sup_{\Phi\in E^*} \inf_{x\in E} \{f(x) + \Phi(x) - \sup \Phi(G)\}. \tag{3.5}$$

Now we shall show, using Theorem 1.2(b) and Theorem 2.1 that the conclusion (3.4′) of Corollary 3.1 (generally, weaker than (3.4)) can be also obtained under the assumption that h is a finite and continuous convex functional on F (without requiring $\inf h(F) > -\infty$). To this end, let us first prove

LEMMA 3.2. *Let $(F \xrightarrow{u} X)$ be a locally convex linear system, Ω a convex subset of X with $u(F) \cap \Omega \neq \varnothing$, and h a finite and continuous convex functional on F. Then*

$$\sup_{0\neq\Phi\in X^*} \inf_{\substack{y\in F\\ \Phi(u(y))\in\Phi(\Omega)}} h(y) = \sup_{\Phi\in X^*} \inf_{y\in F} \{h(y) + \Phi(u(y)) - \sup \Phi(u(F)\cap\Omega)\}. \tag{3.6}$$

Proof. If $\Phi \in X^*$, $\Phi \neq 0$ and $\Phi' \equiv 0$, then, clearly,

$$\inf_{\substack{y\in F\\ \Phi(u(y))\in\Phi(\Omega)}} h(y) \geqslant \inf h(F) = \inf_{y\in F} \{h(y) + \Phi'(u(y)) - \sup \Phi'(u(F)\cap\Omega)\}$$
$$\geqslant \inf_{y\in F} \{h(y) + \Phi(u(y)) - \sup \Phi(u(F)\cap\Omega)\}$$

and hence

$$\sup_{0\neq\Phi\in X^*} \inf_{\substack{y\in F\\ \Phi(u(y))\in\Phi(\Omega)}} h(y) \geqslant \sup_{\Phi\in G^*} \inf_{y\in F} \{h(y) + \Phi(u(y)) - \sup \Phi(u(F)\cap\Omega)\}. \tag{3.7}$$

In order to prove the opposite inequality, let $\Phi \in X^*$, $\Phi \neq 0$. Then, since Ω is convex, $\Phi(\Omega) \subset R$ is an interval (possibly infinite) and hence the set

$$G_\Phi = \{ y \in F \mid \Phi(u(y)) = u^*(\Phi)(y) \in \Phi(\Omega)\} \tag{3.8}$$

is a "strip" in F (possibly infinite). Consequently, for any $\Psi \in F^*$ we have

$$\begin{aligned} \sup \Psi(G_\Phi) &= \sup \lambda_0 \Phi(u(G_\Phi)) &&\text{if } \Psi = \lambda_0 u^*(\Phi) \quad \text{for some } \lambda_0 \in R \\ &= +\infty &&\text{if } \Psi \neq \lambda u^*(\Phi) \quad (\lambda \in R). \end{aligned} \tag{3.9}$$

Therefore, applying Theorem 1.2(b) to the strip G_Φ, and observing that $\{z \in F \mid u(z) \in \Omega\} \subset G_\Phi$, we obtain

$$\begin{aligned} \inf_{\substack{y\in F \\ \Phi(u(y))\in\Phi(\Omega)}} h(y) &= \sup_{\Psi\in F^*} \inf_{y\in F} \{h(y) + \Psi(y) - \sup \Psi(G_\Phi)\} \\ &= \sup_{\lambda_0\in R} \inf_{y\in F} \{h(y) + \lambda_0\Phi(u(y)) - \sup_{\substack{z\in F \\ \Phi(u(z))\in\Phi(\Omega)}} \lambda_0\Phi(u(z))\} \\ &\leqslant \sup_{\lambda_0\in R} \inf_{y\in F} \{h(y) + \lambda_0\Phi(u(y)) - \sup \lambda_0\Phi(u(F) \cap \Omega)\}, \end{aligned} \tag{3.10}$$

whence, since $\Phi \in X^*$ with $\Phi \neq 0$ was arbitrary,

$$\sup_{0\neq\Phi\in X^*} \inf_{\substack{y\in F \\ \Phi(u(y))\in\Phi(\Omega)}} h(y) \leqslant \sup_{\Phi\in X^*} \inf_{y\in F} \{h(y) + \Phi(u(y)) - \sup \Phi(u(F) \cap \Omega)\}. \tag{3.11}$$

From (3.7) and (3.11) it follows that we have (3.6), which completes the proof of Lemma 3.2.

Combining Theorem 2.1 and Lemma 3.2, we obtain

THEOREM 3.3. *Let $(F \to^u X)$ be a locally convex linear system, Ω a convex subset of X with $u(F) \cap \Omega \neq \varnothing$ and h a finite and continuous convex functional on F, such that Ω and the sets Γ_r $(r \in R)$ defined by* (2.2) *are closed for a topology τ on X, weaker than or equal to the initial topology on X, and either Ω or the sets Γ_r $(r \in R)$ are compact for τ. Then we have* (3.4′).

Remark 3.1. (a) In the particular case when $(F \to^u X)$ is a Banach linear system, h is the continuous convex functional on F defined by

$$h(y) = \|y\| \qquad (y \in F), \tag{3.12}$$

τ is the norm topology on X and Ω is a singleton, Theorem 3.1 or 3.3 yields a result of Rolewicz ([3, Theorem V.2.5]). Actually, the proof of Rolewicz, given in [3], shows that *in this particular case the* sup *in* (3.4′) *is attained for some $\Phi = \Phi_0$*. Let us observe that the proof of [3] *remains valid for the situation of Theorem* 3.1, *with Ω a singleton* (*hence compact for τ*) *and Γ_r $(r \in R)$ closed* *for* τ, but apparently it cannot be extended to the situation of Theorem 3.3 with $\Omega \subset X$ an arbitrary convex target set, satisfying $u(F) \cap \Omega \neq \varnothing$.

(b) In the particular case when $F = X$ and $u = I_F$, denoting in Theorem 3.3 $\Omega = G$, $h = f$, we obtain a weaker result than Theorem 1.2(b).

4. Strip Theorems, Hyperplane Theorems, and Lagrangian Theorems of Strong Duality for Linear Systems

Let us first prove the following sharpening of Theorem 1.3(a), which we shall need in the sequel:

Theorem 4.1. *Let E be a locally convex space, $f\colon E \to \bar{R}$ a convex functional and $x_0 \in E$, such that the set*

$$A_{x_0} = \{x \in E \mid f(x) < f(x_0)\} \tag{4.1}$$

is non-empty and open. Then we have

$$f(x_0) = \sup_{0 \neq \Phi \in E^*} \inf_{\substack{x \in E \\ \Phi(x) = \Phi(x_0)}} f(x) \tag{4.2}$$

and there exists $\Phi_0 \in E^*$, $\Phi_0 \neq 0$, such that

$$f(x_0) = \inf_{\substack{x \in E \\ \Phi_0(x) = \Phi_0(x_0)}} f(x) \tag{4.3}$$

(*i.e., for which the* sup *in* (4.2) *is attained*).

Proof. The proof is essentially that of [6, Theorem 2.1], and we give it here for the sake of completeness.

Since $x_0 \notin A_{x_0}$ and since by our assumptions A_{x_0} is non-empty, open and convex, there exists (see, e.g., [1, p. 417, Theorem 8]) a functional $\Phi_0 \in E^*$, $\Phi_0 \neq 0$, such that

$$\Phi_0(x_0) \geqslant \sup \Phi_0(A_{x_0}). \tag{4.4}$$

Clearly,

$$f(x_0) \geqslant \sup_{0 \neq \Phi \in E^*} \inf_{\substack{x \in E \\ \Phi(x) = \Phi(x_0)}} f(x) \geqslant \inf_{\substack{x \in E \\ \Phi_0(x) = \Phi_0(x_0)}} f(x). \tag{4.5}$$

Let us show that in (4.5) the equality signs hold, which will complete the proof. If not, then there exists $x' \in E$ with $\Phi_0(x') = \Phi_0(x_0)$, such that $f(x_0) > f(x')$ (so $x' \in A_{x_0}$). But ,since A_{x_0} is open and $x' \in A_{x_0}$, $\Phi_0 \in E^*$, $\Phi_0 \neq 0$, we must have $\Phi_0(x') < \sup \Phi_0(A_{x_0})$, whence, by (4.4), we obtain $\Phi_0(x') < \Phi_0(x_0)$, which contradicts the definition of x'. This completes the proof of Theorem 4.1.

LEMMA 4.1. *Let F, X be two locally convex spaces, $u: F \to X$ a mapping, Ω a subset of X with $u(F) \cap \Omega \neq \varnothing$ and $h: F \to \bar{R}$ a functional, such that*

$$\inf h(F) < \inf_{\substack{y \in F \\ u(y) \in \Omega}} h(y) = a \tag{4.6}$$

and that either Ω or the set

$$\Theta_a = u(\{y \in F \mid h(y) < a\}) \tag{4.7}$$

is open. If $f: X \to \bar{R}$ is the functional defined by

$$f(x) = \inf_{\substack{y \in F \\ u(y) \in x + \Omega}} h(y) \qquad (x \in X), \tag{4.8}$$

then the set

$$A_0 = \{x \in E \mid f(x) < a = f(0)\} \tag{4.9}$$

is non-empty and open.

Proof. By (4.6), the set (4.9) is non-empty (if $y \in F$, $h(y) < a$, then any $x \in u(y) - \Omega$ belongs to (4.9)).

Now let $x \in E$, $f(x) < a$. Then, by (4.8), there exist $y_0 \in F$ and $\omega_0 \in \Omega$ such that $x = u(y_0) - \omega_0$ and $h(y_0) < a$. If Ω is open, then $u(y_0) - \Omega$ is an open set containing x and for any $x' = u(y_0) - \omega' \in u(y_0) - \Omega$ we have

$$f(x') = \inf_{\substack{y \in F \\ u(y) \in x' + \Omega}} h(y) \leqslant h(y_0) < a.$$

On the other hand, if $\Theta_a = u(\{y \in F \mid h(y) < a\})$ is open, then $\Theta_a - \omega_0$ is an open set containing x and for any $x' = u(y') - \omega_0 \in \Theta_a - \omega_0$ we have

$$f(x') = \inf_{\substack{y \in F \\ u(y) \in x' + \Omega}} h(y) \leqslant h(y') < a,$$

which completes the proof of Lemma 4.1.

THEOREM 4.2. *Let $(F \to^u X)$ be a locally convex linear system, Ω a convex subset of X with $u(F) \cap \Omega \neq \varnothing$ and $h: F \to \bar{R}$ a convex functional satisfying* (4.6), *such that either Ω or the set Θ_a (defined by* (4.7)) *is open. Then we have* (2.4) *and there exists $\Phi_0 \in X^*$, $\Phi_0 \neq 0$, such that*

$$\inf_{\substack{y \in F \\ u(y) \in \Omega}} h(y) = \inf_{\substack{y \in F \\ \Phi_0(u(y)) \in \Phi_0(\Omega)}} h(y) \tag{4.10}$$

(*i.e., such that the* sup *in* (2.4) *is attained for $\Phi = \Phi_0$*).

Proof. Define a functional $f: X \to \bar{R}$ by (4.8). Then, by Lemma 2.1, f is a convex functional on X, with $\operatorname{dom}(f) \subset u(F) - \Omega$. Also, by our assumption, (4.6) holds and either Ω or Θ_a is open. Hence, by Lemma 4.1 and Theorem 4.1, there exists $\Phi_0 \in X^*$, $\Phi_0 \neq 0$, such that

$$\inf_{\substack{y\in F\\ u(y)\in\Omega}} h(y) = f(0) = \inf_{\substack{x\in X\\ \Phi_0(x)=0}} f(x) \leqslant \inf_{\substack{x\in u(F)-\Omega\\ \Phi_0(x)=0}} \inf_{\substack{y\in F\\ u(y)\in x+\Omega}} h(y)$$
$$= \inf_{\substack{y\in F\\ \Phi_0(u(y))\in\Phi_0(\Omega)}} h(y)$$

(the last equality follows from (2.6)). This, together with the obvious inequality $\geqslant$ in (2.4) yields (2.4) and (4.10), completing the proof of Theorem 4.2.

Remark 4.1. (a) In particular, if u is an open continuous linear mapping of F onto X and h is also finite and continuous on F, and if (4.6) holds, then the set Θ_a defined by (4.7) is nonempty and open and thus we can apply Theorem 4.2. Moreover, in this case the assumption (4.6) can be omitted, as we shall show in Theorem 4.4(a) below.

(b) In the particular case when $(F \to^u X)$ is a Banach linear system and *h is also finite and continuous on F*, Rolewicz has given, with the indication of a different proof, the following result ([4, Theorem 5.2]; however, the assumption $u(F) \cap \Omega \neq \varnothing$ and the conclusion $\Phi_0 \neq 0$ should be added there): *Under the above-mentioned assumptions, if we have* (4.6) *and if either* $\operatorname{Int} \Omega \neq \varnothing$ *or* $\operatorname{Int} \Gamma_a \neq \varnothing$ *(where Γ_a is defined by* (2.2)), *then the conclusions of Theorem* 4.2 *hold*. One can show that this result remains valid for any locally convex linear system $(F \to^u X)$.

It is worth while to mention separately the particular case $F = X$, $u = I_F$, of Theorem 4.2. In this case, denoting $F = X = E$, $\Omega = G$, $h = f$ in Theorem 4.2, we obtain

THEOREM 4.3. *Let E be a locally convex space, G a convex subset of E and $f: E \to \bar{R}$ a convex functional, such that*

$$\inf f(E) < \inf f(G) = a \tag{4.11}$$

and that either G or the set

$$C_a = \{x \in E \mid f(x) < a\} \tag{4.12}$$

is open. Then we have (2.12) *and there exists $\Phi_0 \in E^*$, $\Phi_0 \neq 0$, such that*

$$\inf f(G) = \inf_{\substack{x\in E\\ \Phi_0(x)\in\Phi_0(G)}} f(x) \tag{4.13}$$

(*i.e., such that the* sup *in* (2.12) *is attained for* $\Phi = \Phi_0$).

Remark 4.2. In the particular case when $F = X$, $u = I_F$, Remark 4.1(b) yields the following result: *Let E be a locally convex space, G a convex subset of E and f a finite and continuous convex functional on E, satisfying* (4.11) *and such that either* $\operatorname{Int} G \neq \varnothing$ *or* $\operatorname{Int}\{x \in E \mid f(x) \leqslant a\} \neq \varnothing$. *Then the conclusions of Theorem* 4.3 *hold.*

Let us return now to the particular case of Theorem 4.2, mentioned in Remark 4.1(a) above.

LEMMA 4.2. *Let $(F \to^u X)$ be a locally convex linear system, such that u is an open continuous linear mapping of F onto X, let Ω be a convex subset of X and let h be a finite and continuous convex functional on F. Then the functional f on X, defined by* (4.8), *is finite and continuous on X.*

Proof. Since $u(F) = X$ and since h is finite, for the functional $f\colon X \to \bar{R}$ defined by (4.8) we have $\operatorname{dom}(f) = u(F) - \Omega = X$; also, by Lemma 2.1, f is convex. Furthermore, since u is open and continuous and since h is finite and continuous, for any $r > \inf h(F)$ the set $\Theta_r = u(\{y \in F \mid h(y) < r\})$ is non-empty and open. But, for any $y_0 \in F$ with $h(y_0) < r$ and any $\omega_0 \in \Omega$ we have

$$f(u(y_0) - \omega_0) = \inf_{\substack{y \in F \\ u(y) \in u(y_0) - \omega_0 + \Omega}} h(y) \leqslant h(y_0) < r,$$

so $\Theta_r - \omega_0 \subset \{x \in E \mid f(x) < r\}$. Thus, f is bounded on the non-empty open set $\Theta_r - \omega_0$ and hence (see, e.g., [2, Theorem (6.2.7)]), f is continuous on $\operatorname{Int} \operatorname{dom}(f) = X$. This completes the proof of Lemma 4.2.

THEOREM 4.4. *Let $(F \to^u X)$ be a locally convex linear system, such that u is an open continuous linear mapping of F onto X, let Ω be a convex subset of X and let h be a finite and continuous convex functional on F. Then*

(a) *We have* (2.4) *and there exists* $\Phi_0 \in X^*$, $\Phi_0 \neq 0$ *satisfying* (4.10).

(b) *When* (4.6) *holds, for a functional* $\Phi_0 \in X^*$, $\Phi_0 \neq 0$, we have (4.10) *if and only if there exists* $\alpha_0 \in R$ *such that*

$$\inf_{\substack{y \in F \\ u(y) \in \Omega}} h(y) = \inf_{y \in F} \{h(y) + \alpha_0 \Phi_0(u(y)) - \sup \alpha_0 \Phi_0(\Omega)\} \tag{4.14}$$

(*i.e., such that the first* sup *in* (3.4) *is attained for* $\Phi = \alpha_0 \Phi_0$).

Proof. (a) By Lemma 4.2 and Theorem 1.3(a), applied to $f(0) = a$, we obtain, as in the above proof of Theorem 4.2, that we have (2.4) and there exists $\Phi_0 \in X^*$, $\Phi_0 \neq 0$, satisfying (4.10).

(b) Let $\Phi_0 \in X^*$, $\Phi_0 \neq 0$. Then, as shown by the above proof of part (a), (4.10) is equivalent to

$$f(0) = \inf_{\substack{x \in X \\ \Phi_0(x)=0}} f(x), \tag{4.15}$$

where f is defined by (4.8), and f is a finite and continuous convex functional on X. Thus, (b) follows from Theorem 1.3(b) and Lemma 3.1 (applied to $\Phi = \alpha_0\Phi_0$), which completes the proof of Theorem 4.4.

COROLLARY 4.1. *Assume that* $(F \to^u X)$, Ω *and* h *are as in Theorem* 4.2 *or* 4.4. *Then*

(a) *For any* $\Phi_0 \in X^*$, $\Phi_0 \neq 0$, *satisfying* (4.10), *we have*

$$\inf_{\substack{y \in F \\ u(y) \in \Omega}} h(y) = \inf_{\substack{y \in F \\ \Phi_0(u(y)) \in \Phi_0(u(F) \cap \Omega)}} h(y). \tag{4.10'}$$

(b) *For any* $\Phi_0 \in X^*$, $\Phi_0 \neq 0$ *and* $\alpha_0 \in R$ *satisfying* (4.14), *we have*

$$\inf_{\substack{y \in F \\ u(y) \in \Omega}} h(y) = \inf_{y \in F}\{h(y) + \alpha_0\Phi_0(u(y)) - \sup \alpha_0\Phi_0(u(F) \cap \Omega)\}. \tag{4.14'}$$

Proof. The proofs of (a) and (b) are similar, respectively, to those of Corollaries 2.1 and 3.1 above.

In the particular case when u is an open continuous linear mapping of F onto X, h is a finite and continuous convex functional on F and $\Omega = \{x_0\}$, a singleton, from Theorem 4.4 we obtain the following result (the first part of which, in the case when $(F \to^u X)$ is a Banach linear system, has been proved, with a different argument, by Rolewicz in [4, Theorem 3.1], under the additional—but superfluous—assumption (4.6) with $\Omega = \{x_0\}$; also, the conclusion $\Phi_0 \neq 0$ should be added there):

COROLLARY 4.2. *Let* $(F \to^u X)$ *be a locally convex linear system, such that* u *is an open continuous linear mapping of* F *onto* X, *let* $x_0 \in u(F) = X$ *and let* h *be a finite and continuous convex functional on* F. *Then*

(a) *There exists* $\Phi_0 \in X^*$, $\Phi_0 \neq 0$, *such that*

$$\inf_{\substack{y \in F \\ u(y)=x_0}} h(y) = \inf_{\substack{y \in F \\ \Phi_0(u(y))=\Phi_0(x_0)}} h(y). \tag{4.16}$$

(b) *When* (4.6) *holds, for a functional* $\Phi_0 \in X^*$, $\Phi_0 \neq 0$, we have (4.16) *if and only if there exists* $\alpha_0 \in R$ *such that*

$$\inf_{\substack{y \in F \\ u(y)=x_0}} h(y) = \inf_{y \in F}\{h(y) + \alpha_0\Phi_0(u(y)) - \alpha_0\Phi_0(x_0)\}. \tag{4.17}$$

Remark 4.3. (a) In this particular case, the functional f defined in Lemma 4.1 (by formula (4.8)) reduces to (2.8), hence (2.9) (since $u(F) = X$). Similarly to Remark 2.2, in this case one can also consider, instead of f, the primal functional f_0 defined by (2.10); then, once the continuity of f_0 is established (similarly to the proof of Lemma 4.2), one can apply Theorem 1.3(a) for $E = u(F)$ and the Hahn–Banach theorem, to obtain the existence of a functional $\Phi_0 \in X^*$, $\Phi_0 \neq 0$, satisfying

$$\inf_{\substack{y \in F \\ u(y)=x_0}} h(y) = f_0(x_0) = \inf_{\substack{x \in u(F) \\ \Phi_0(x)=\Phi_0(x_0)}} f_0(x) = \inf_{\substack{x \in u(F) \\ \Phi_0(x)=\Phi_0(x_0)}} \inf_{\substack{y \in F \\ u(y)=x}} h(y)$$

$$= \inf_{\substack{y \in F \\ \Phi_0(u(y))=\Phi_0(x_0)}} h(y),$$

that is, (4.16). Corollary 4.2(b) follows similarly, using Theorem 1.3(b). Let us also mention, for this case, the following alternative proof of the continuity of the primal functional f_0 defined by (2.10): Since u is an open mapping of F onto X, $F/\mathrm{Ker}\, u$ is isomorphic to X, by the induced mapping

$$\tilde{u}(y + \mathrm{Ker}\, u) = u(y) \qquad (y + \mathrm{Ker}\, u \in F/\mathrm{Ker}\, u) \tag{4.18}$$

(see, e.g., [5, Chap. III, Proposition 1.2]). Furthermore, since h is continuous and convex, the functional $\tilde{h}$ on $F/\mathrm{Ker}\, u$ defined by

$$\tilde{h}(y + \mathrm{Ker}\, u) = \inf_{y' \in y+\mathrm{Ker}\, u} h(y') \qquad (y + \mathrm{Ker}\, u \in F/\mathrm{Ker}\, u) \tag{4.19}$$

is continuous (see, e.g., [12]). But, by (2.10), (4.19), and (4.18) we have

$$f_0(u(y)) = \inf_{\substack{y' \in F \\ u(y')=u(y)}} h(y') = \inf_{y' \in y+\mathrm{Ker}\, u} h(y') = \tilde{h}(y + \mathrm{Ker}\, u)$$

$$= \tilde{h}(\tilde{u}^{-1}\tilde{u}(y + \mathrm{Ker}\, u)) = \tilde{h}\tilde{u}^{-1}(u(y)) \qquad (u(y) \in u(F) = X),$$

that is, $f_0 = \tilde{h}\tilde{u}^{-1}$, and hence f_0 is continuous. This completes the proof.

(b) In the particular case when $(F \rightarrow^u X)$ is a Banach linear system (hence, by the open mapping theorem, every continuous linear mapping u of F onto X is open), Corollary 4.2(a) has been proved, with a different argument, by Rolewicz ([4, Theorem 3.1]), under the additional—but superfluous—assumption (4.6) with $\Omega = \{x_0\}$ (also, the conclusion $\Phi_0 \neq 0$ should be added in [4]). Let us also recall that if $(F \rightarrow^u X)$ is a locally convex linear system and h a finite and continuous convex functional on F, $(F \rightarrow^u X)$ is said to *satisfy the Pontryagin maximum principle with respect to* h, or, briefly, $\{(F \rightarrow^u X), h\}$ is said to *satisfy the* PMP, provided that for each $x_0 \in u(F)$ there exists $\Phi_0 \in X^*$ (not necessarily $\neq 0$) such that (4.16) holds (see [11]; for the particular case of Banach linear systems, see [4, Sect. 3]). Similarly, if Ω is a non-empty convex subset of X with $u(F) \cap \Omega \neq \varnothing$, we shall say that $(F \rightarrow^u X)$ *satisfies the*

Pontryagin maximum principle with respect to the target set Ω *and the functional* h or, briefly, that $\{(F \to^u X), \Omega, h\}$ *satisfies the* PMP, if there exists $\Phi_0 \in X^*$ (not necessarily $\neq 0$) such that (4.10) holds (for the particular case of Banach linear systems, see [4, Sect. 5]).

5. Some Counter-Examples

Using the functional (2.10), one can give an example of a Banach space E showing that in Theorem 1.3(a) the assumption of continuity of f cannot be replaced by the weaker assumption of lower sem-continuity of f and that the sup in (1.6) of Theorem 1.1 need not be attained (such an example, with E only a locally convex space, has been given in [8, Example 2.1]). Also, this example will show that the sup in (2.7) of Corollary 2.2 (and hence in (2.4) of Theorem 2.1) need not be attaained.

Example 5.1. Let $F = X = E = l^2$ and let

$$u(y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \varphi_i(y)\, e_i \qquad (y \in F), \tag{5.1}$$

where $\{e_n\}$ is the sequence of unit vectors in X and $\{\varphi_n\}$ the sequence of coordinate functionals on F, so u is continuous, linear and one-to-one. Furthermore, let

$$h(y) = \|y\| \qquad (y \in F), \tag{5.2}$$

so h is a continuous convex functional on F, and define a convex functional f_0 on X by (2.10). Then for each $r \in R$ the set $\{y \in F \mid h(y) \leqslant r\}$ is weakly compact (empty, if $r < 0$), whence, since u is continuous, each Γ_r defined by (2.2) is weakly compact, and thus closed (in the norm topology). Consequently, by the proof of Lemma 2.2, f_0 is lower semi-continuous on X and hence we have (1.6) with $f = f_0$. However, $\overline{u(F)} = X \neq u(F)$ and therefore, by a result of Rolewicz ([4, Theorem 3.3], where the assumption that each Γ_r is closed, should be added), $(F \to^u X)$ does not satisfy the PMP, so there exists $x_0 \in u(F)$ for which there is no $\Phi_0 \in X^* = E^*$ satisfying (4.16). Then, for this x_0, the sup in (1.6) is not attained. Let us also observe that, since u is one-to-one, we have

$$f_0(x) = \| u^{-1}(x) \| \qquad (x \in u(F)), \tag{5.3}$$

and hence, since $\overline{u(F)} = X \neq u(F)$, f_0 is discontinuous at each $x_0 \in u(F) = \mathrm{dom}(f_0)$.

Using again the functional (2.10), one can give an example of a normed linear space E showing that in Theorem 1.3(a) the assumption of continuity of f, even at one point, is not necessary (such an example, with E only a locally convex space and with f lower semi-continuous, has been given in [8, Example 3.1]). Moreover, this example will also show that in Theorem 1.1 and 1.3(a) even the lower semi-continuity of f is not necessary.

EXAMPLE 5.2. Let $F = X = c_0$ and define again u, h and f_0 by (5.1), (5.2), and (2.10), respectively. Then, as we have shown in [10, Example 2.1], $\{(F \to^u X), h\}$ satisfies the PMP and hence, by the obvious inequality $\geqslant$ in (1.6), the conclusions of Theorems 1.1 and 1.3(a) hold for f_0 on $E = u(F)$. However, by the argument of Example 5.1 above, f_0 is discontinuous at each $x_0 \in E$. Moreover, let us show that f_0 is not lower semi-continuous on E. Indeed, the set $\Gamma_1 = \{u(y) \mid y \in F, \|y\| \leqslant 1\}$ is not closed (this follows from the facts that $\{(F \to^u X), h\}$ satisfies the PMP and $\overline{u(F)} \neq u(F)$, combined with [4, Theorem 3.3], but can be seen also directly). But, since now (5.3) holds and $E = u(F)$, we have

$$\begin{aligned}\{x \in E \mid f_0(x) \leqslant 1\} &= \{x \in E \mid \|u^{-1}(x)\| \leqslant 1\} \\ &= \{u(y) \mid y \in F, \|u^{-1}(u(y))\| \leqslant 1\} = \Gamma_1, \qquad (5.4)\end{aligned}$$

and hence f_0 is not lower semi-continuous on E.

Remark 5.1. Similarly, one can see that the conclusions of Theorem 4.2, Remark 4.1(b), Theorem 4.4(a), and Corollary 4.2(a) hold for $\Omega = \{x_0\}$ and f defined by (4.8) (that is, by (2.9)), although the sets Ω and $\Theta_a = \{u(y) \mid y \in F, \|y\| < a\}$ are not open (also, Int $\Omega = \varnothing$, Int $\Gamma_a = \varnothing$). Thus, the assumptions on Ω or Θ_a or Γ_a in these results are not necessary either.

Let us also mention the following simple example of a finite-dimensional Banach space E (hence f is continuous on Int dom(f)), having some similar properties and showing, in addition, that the conclusions of Theorems 1.1 and 1.3(a) do not imply the conclusion of Theorem 1.2(a).

EXAMPLE 5.3. Let $E = R^2$ (the euclidean plane) and for $x = (\xi_1, \xi_2) \in E$ let

$$\begin{aligned} f(x) &= \xi_1^2 + \xi_2^2 && \text{if } \xi_1^2 + \xi_2^2 < 1 \\ &= 2 && \text{if } \xi_1^2 + \xi_2^2 = 1 \qquad (5.5) \\ &= +\infty && \text{if } \xi_1^2 + \xi_2^2 > 1. \end{aligned}$$

Then f is a proper convex functional on E, continuous on $\{x = (\xi_1, \xi_2) \in E \mid \xi_1^2 + \xi_2^2 < 1\} = \text{Int dom}(f)$, but not even lower semi-continuous at any point x_0 of $\{x = (\xi_1, \xi_2) \mid \xi_1^2 + \xi_2^2 = 1\} \subset \text{dom}(f)$, and thus $f(x_0) \neq f^{**}(x_0)$ at any such point x_0. Hence, as shown by the converse argument to the proof of Theorem 1.2(a), given in [9], the conclusion (1.7) of Theorem 1.2(a) does not hold at any such point x_0. However, by Theorem 4.1 (or, checking directly), the conclusions of Theorem 1.1 and 1.3(a) hold for all $x_0 \neq 0$ (although the assumptions of these theorems are not satisfied, as was observed above) and, clearly, also for $x_0 = 0$.

6. Duality Theorems for Convex Systems

Now we shall apply our methods of Sections 2–4 to convex systems. In the case of convex systems, the role of Lemma 2.1 will be played by the following well-known observation:

Lemma 6.1. *Let F be a linear space, X a partially ordered linear space, u a convex mapping of F into X, Ω the convex cone of all non-positive elements in X, and $h: F \to \bar{R}$ a convex functional. Then the functional $f: X \to \bar{R}$, defined by*

$$f(x) = \inf_{\substack{y \in F \\ u(y) \in x + \Omega}} h(y) = \inf_{\substack{y \in F \\ u(y) \leqslant x}} h(y) \qquad (x \in X), \tag{6.1}$$

is convex.

We recall that if X is a partially ordered locally convex space, a functional $\Phi \in X^*$ is said to be *non-negative* if $\Phi(x) \geqslant 0$ for all $x \in X$, $x \geqslant 0$. We shall write $\Phi > 0$ if $\Phi \geqslant 0$ and $\Phi \neq 0$.

Theorem 6.1. *Let $(F \to^u X)$ be a convex system, such that the convex cone Ω of all non-positive elements in X satisfies $u(F) \cap \Omega \neq \varnothing$, and $h: F \to \bar{R}$ a convex functional, such that the sets Γ_r $(r \in R)$ defined by*

$$\Gamma_r = u(\{y \in F \mid h(y) \leqslant r\}) \qquad (r \in R) \tag{6.2}$$

are weakly compact. Then we have

$$\inf_{\substack{y \in F \\ u(y) \leqslant 0}} h(y) = \sup_{0 < \Phi \in X^*} \inf_{\substack{y \in F \\ \Phi(u(y)) \leqslant 0}} h(y). \tag{6.3}$$

Proof. Since Ω is weakly closed, by the above proof of Theorem 2.1 (using Lemma 6.1 instead of Lemma 2.1) we have (2.4). But, since Ω is the non-positive cone in X, there holds

$$\begin{aligned} \Phi(\Omega) &= (-\infty, 0] && \text{if} \quad 0 < \Phi \in X^* \\ &= [0, +\infty) && \text{if} \quad 0 > \Phi \in X^* \\ &= R && \text{if} \quad 0 \neq \Phi \in X^*, \quad 0 \not< \Phi, \quad 0 \not> \Phi, \end{aligned} \tag{6.4}$$

whence

$$\begin{aligned}
&\{y \in F \mid \Phi(u(y)) \in \Phi(\Omega)\} \\
&\quad = \{y \in F \mid \Phi(u(y)) \leqslant 0\} && \text{if } \ 0 < \Phi \in X^*, \\
&\quad = \{y \in F \mid (-\Phi)(u(y)) \leqslant 0\} && \text{if } \ 0 > \Phi \in X^* \quad (\Leftrightarrow 0 < -\Phi \in X^*), \\
&\quad = F && \text{if } \ 0 \neq \Phi \in X^*, \quad 0 \not< \Phi, \quad 0 \not> \Phi,
\end{aligned} \tag{6.5}$$

and thus (2.4) coincides now with (6.3). This completes the proof of Theorem 6.1.

Remark 6.1. The sets $\{y \in F \mid \Phi(u(y)) \leqslant 0\}$ in (6.3) need not be "strips".

THEOREM 6.2. *Let* $(F \to^u X)$ *be a convex system, such that the convex cone* Ω *of all non-positive elements in* X *satisfies* $u(F) \cap \Omega \neq \varnothing$, *and* $h: F \to (-\infty, +\infty]$ *a convex functional with* $\inf h(F) > -\infty$, *such that the sets* Γ_r $(r \in R)$ *defined by* (6.2) *are weakly compact. Then*

$$\inf_{\substack{y \in F \\ u(y) \leqslant 0}} h(y) = \sup_{0 \leqslant \Phi \in X^*} \inf_{y \in F} \{h(y) + \Phi(u(y))\}. \tag{6.6}$$

Proof. Since Ω is weakly closed, by the above proof of Theorem 3.1 (using Lemma 6.1 instead of Lemma 2.1), we have (3.4). But, since Ω is the non-positive cone in X, there holds (6.4) above, whence

$$\begin{aligned}
\sup \Phi(\Omega) &= 0 && \text{if } \ 0 \leqslant \Phi \in X^* \\
&= +\infty && \text{if } \ 0 \not\leqslant \Phi \in X^*,
\end{aligned} \tag{6.7}$$

and thus (3.4) coincides now with (6.6). This completes the proof of Theorem 6.2.

Remark 6.2. Theorem 6.2 may be regarded as a complement to the usual Kuhn–Tucker theorem, which says that under Slater's constraint qualification (i.e., under the assumption that there exists $y \in F$ with $u(y) \in \operatorname{Int} \Omega$), and if $h(F) \subset R$, $\inf_{y \in F, u(y) \leqslant 0} h(y) > -\infty$, we have (6.6) and there exists $\Phi_0 \in X^*$, $\Phi_0 > 0$, for which the sup in (6.6) is attained.

Finally, let us give

THEOREM 6.3. *Let* $(F \to^u X)$ *be a convex system, such that the convex cone* Ω *of all non-positive elements in* X *satisfies* $u(F) \cap \Omega \neq \varnothing$, *and* $h: F \to \bar{R}$ *a convex functional, such that*

$$\inf h(F) < \inf_{\substack{y \in F \\ u(y) \leqslant 0}} h(y) = a \tag{6.8}$$

and that the set

$$\Theta_a = u(\{y \in F \mid h(y) < a\}) \tag{6.9}$$

is open. Then we have (6.3) *and there exists* $\Phi_0 \in X^*$, $\Phi_0 > 0$, *such that*

$$\inf_{\substack{y \in F \\ u(y) \leqslant 0}} h(y) = \inf_{\substack{y \in F \\ \Phi_0(u(y)) \leqslant 0}} h(y) \tag{6.10}$$

(*i.e., such that the sup in* (6.3) *is attained for* $\Phi = \Phi_0$).

Proof. By the above proof of Theorem 4.2 (using Lemma 6.1 instead of Lemma 2.1), we have (2.4) and there exists $\Phi_0 \in X^*$, $\Phi_0 \neq 0$, satisfying (4.10). But, by the above proof of Theorem 6.1, (2.4) coincides with (6.3). Furthermore, by (6.5) we have

$$\begin{aligned} \inf_{\substack{y \in F \\ \Phi_0(u(y)) \in \Phi_0(\Omega)}} h(y) &= \inf_{\substack{y \in F \\ \Phi_0(u(y)) \leqslant 0}} h(y) && \text{if } 0 < \Phi_0 \\ &= \inf_{\substack{y \in F \\ (-\Phi_0)(u(y)) \leqslant 0}} h(y) && \text{if } 0 > \Phi_0 \; (\Leftrightarrow 0 < -\Phi_0) \\ &= \inf h(F) && \text{if } 0 \not< \Phi_0, \quad 0 \not> \Phi_0, \end{aligned} \tag{6.11}$$

and thus, by (6.8), formula (4.10) coincides now with (6.10) for $\Phi_0 > 0$ or $-\Phi_0 > 0$. This completes the proof of Theorem 6.3.

Note added in proof. (1) The primal functional f_0 defined by (2.10) can be also used to study the general optimization problem (1.1), observing that

$$\inf_{\substack{y \in F \\ u(y) \in \Omega}} h(y) = \inf f_0(\Omega).$$

(2) The equality sign in (3.6) should be replaced by $\leqslant$ and the sentence containing (3.7) should be deleted. Then, to prove Theorem 3.3, combining Theorem 2.1 and Lemma 3.2 we obtain the inequality $\leqslant$ in (3.4′), while the inequality $\geqslant$ in (3.4′) is obvious (see the proof of Corollary 3.1). Alternatively, one can obtain equality in (3.6), replacing there (and in the proof) sup $\Phi(\Omega)$ by sup $\Phi(u(F) \cap \Omega)$; this yields again (3.4′).

(3) Every Lagrangian duality theorem implies a strip theorem, and Theorem 6.2 implies a particular case of Theorem 6.1, by the obvious inequalities

$$\inf f(G) \geqslant \inf_{\substack{x \in E \\ \Phi(x) \in \Phi(G)}} f(x) \geqslant \inf_{\substack{x \in E \\ \Phi(x) \in \Phi(G)}} f(x) \geqslant \inf_{y \in E} \{f(x) + \Phi(x) - \sup \Phi(G)\} \quad (\Phi \in E^*),$$

$$\inf_{\substack{y \in F \\ u(y) \in \Omega}} h(y) \geqslant \inf_{\substack{y \in F \\ \Phi(u(y)) \in \Phi(\Omega)}} h(y) \geqslant \inf_{y \in F} \{h(y) + \Phi(u(y)) - \sup \Phi(\Omega)\} \quad (\Phi \in X^*),$$

$$\inf_{\substack{y \in F \\ u(y) \leqslant 0}} h(y) \geqslant \inf_{\substack{y \in F \\ \Phi(u(y)) \leqslant 0}} h(y) \geqslant \inf_{y \in F} \{h(y) + \Phi(u(y))\} \quad (0 \leqslant \Phi \in X^*),$$

respectively.

REFERENCES

1. N. DUNFORD AND J. SCHWARTZ, "Linear Operators," Part I: "General Theory," Pure and Appl. Math. No. 7, Interscience, New York/London, 1958.
2. P.-J. LAURENT, "Approximation et optimisation," Hermann, Paris, 1972.
3. S. ROLEWICZ, "Analiza funkcjonalna i teoria sterowania," PWN, Warsaw, 1974.
4. S. ROLEWICZ, On general theory of linear systems, *Beiträge zur Anal.* **8** (1976), 119–127.
5. H. H. SCHAEFER, "Topological Vector Spaces," Macmillan, New York, 1966.
6. I. SINGER, Generalizations of methods of best approximation to convex optimization in locally convex spaces. II: Hyperplane theorems, *J. Math. Anal. Appl.* **69** (1979), 571–584.
7. I. SINGER, Some new applications of the Fenchel–Rockafellar duality theorem: Lagrange multiplier theorems and hyperplane theorems for convex optimization and best approximation, *Nonlinear Anal. Theory Methods Appl.* **3** (1979), 239–248.
8. I. SINGER, Maximization of lower semi-continuous convex functionals on bounded subsets of locally convex spaces. I: Hyperplane theorems, *Appl. Math. Optimiz.* **5** (1979), 349–362.
9. I. SINGER, Maximization of lower semi-continuous convex functionals on bounded subsets of locally convex spaces. II: Quasi-Lagrangian duality theorems. *Result. Math.*, in press.
10. I. SINGER, On the Pontryagin maximum principle for constant-time linear control systems in Banach spaces, *J. Optimization Theory Appl.* **27** (1979), 325–331.
11. I. SINGER, A characterization of constant-time linear control systems satisfying the Pontryagin maximum principle. *J. Optimization Theory Appl.* **32** (1980), in press.
12. M. WRIEDT, Stetigkeit von Optimierungsoperatoren. *Arch. Math.* **28** (1977), 652–656.